On the covering radius problem for the lattices

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1 Introduction

1.1 Some definitions from lattice theory

Let $\mathbb{Z}$ be the ring of rational integers and $\mathbb{R}$ the field of real numbers. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be linearly independent vectors over $\mathbb{R}$in $\mathbb{R}^n$. The $\mathbb{Z}$module generated by $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is called a lattice L in $\mathbb{R}^n$. These vectors are called a basis of the lattice L. The inner product and the norm are defined in L as a subset of $\mathbb{R}^n$.
A lattice L is integral if L satisfies $(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}$ for any $\mathbf{x}, \mathbf{y} \in L$ where (,) is the bilinear form associated to the metric. Two integral lattices L_1 and L_2 are said to be isometric if and only if there exists a bijective linear mapping from L_1 to L_2 preserving the metric. The maximal number of linearly independent vectors over $\mathbb{R}$ in L is called the rank of L. The dual lattice $L^{\#}$ of L is defined by

$$L^{\#} = \{\mathbf{y} \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid (\mathbf{x}, \mathbf{y}) \in \mathbb{Z}, \forall \mathbf{x} \in L\}.$$

Here $\mathbb{Q}$ is the field of rational numbers. A lattice L is even if any element $\mathbf{x}$ of L has even norm $(\mathbf{x}, \mathbf{x})$. In an even lattice L, we say that $\mathbf{x}$ is a $2m$-vector if $(\mathbf{x}, \mathbf{x}) = 2m$ holds for some natural number m. Let $\Lambda_{2m}(L)$ be the set defined by

$$\Lambda_{2m}(L) = \{\mathbf{x} \in L \mid (\mathbf{x}, \mathbf{x}) = 2m\}. \tag{1.1}$$

A lattice L is called unimodular if $L = L^{\#}$. Even unimodular lattices exist only when $n \equiv 0 \pmod 8$. The minimal norm of a lattice is $\mathrm{Min}(L) = \min_{\mathbf{x} \in L \setminus \{0\}} (\mathbf{x}, \mathbf{x})$. When L is even unimodular of rank n it holds that (conf. [31])

$$\mathrm{Min}(L) \leq 2 \left[\frac{n}{24}\right] + 2.$$

Such a lattice which attains the above maximum is said to be extremal.

1.2 The formulation of the problem

When we put a sphere $S_R(\mathbf{x})$ of radius R with the center at each lattice point $\mathbf{x}$ of a given lattice $L \subset \mathbb{R}^n$. If R is large enough, then the set $\bigcup_{\mathbf{x} \in L} S_R(\mathbf{x})$ covers $\mathbb{R}^n$. Therefore we may seek to find the least value R such that

$$\bigcup_{\mathbf{x} \in L} S_R(\mathbf{x}) = \mathbb{R}^n$$

holds. We call such $R = \rho(L)$ the covering radius of the lattice L.

1.3 The simplest non-trivial case. $n = 2$

This case was settled by R. Kershner [22]. He showed that the most efficient lattice covering is the hexagonal lattice covering. His original work is rather complicated and isolated from the methods used in the $n \geq 3$ dimensions.

1.4 Fundamental Parallelepiped

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of L. The point set defined by

$$\mathcal{FP} = \{(a_1 \dot{\mathbf{u}}_1 + a_2 \cdot \mathbf{u}_2 + \cdots + \cdots + a_n \mathbf{u}_n) | 0 \leq a_i \leq 1, i = 1, 2, \ldots, n\}$$

is called a fundamental parallelepiped with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.
From the linear algebra (c.f. for instance I. Satake "Linear Algebra") it is known that the volume $Vol(\mathcal{FP})$ of $\mathcal{FP}$ is the absolute value of the determinant

$$\det(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n).$$

Another formulation of $Vol(\mathcal{FP})$ is to use the Gram matrix of the lattice.

$$Gram(L) = ((\mathbf{u}_i, \mathbf{u}_j))_{1 \leq i,j \leq n}.$$

$$Vol(\mathcal{FP}) = \sqrt{\det(Gram(L))}.$$

2 The Dirichlet-Voronoi region of the lattice

Let L be a lattice in $\mathbb{R}^n$. Let $\mathbf{u}$ be a lattice point other than $\mathbf{0}$. Let $\mathcal{H}_{1/2\mathbf{u}}$ be the hyperplane perpendicular to $\mathbf{u}$ that crosses with $\mathbf{u}$ at the point $1/2\mathbf{u}$. The hyperplane divides the total space $\mathbb{R}^n$ into two half-spaces. Let $\mathcal{H}^+_{1/2\mathbf{u}}(\mathbf{0}, L)$ one of the half-spaces that contains $\mathbf{0}$ plus the hyperplane $\mathcal{H}_{1/2\mathbf{u}}$. The defining equation of $\mathcal{H}_{1/2\mathbf{u}}$ is given by

$$(\mathbf{x}, \mathbf{x}) = (1/2\mathbf{u}, 1/2\mathbf{u}) + (\mathbf{x} - 1/2\mathbf{u}, \mathbf{x} - 1/2\mathbf{u}).$$

This is simply the Pithagorian Theorem. The above equation can be rewritten as

$$(\mathbf{x}, \mathbf{u}) = 1/2(\mathbf{u}, \mathbf{u}). \tag{2.1}$$

Consequently the definig inequality of $\mathcal{H}^+_{\frac{1}{2}\mathbf{u}}(\mathbf{0}, L)$ is given by

$$(\mathbf{x}, \mathbf{u}) \leq 1/2(\mathbf{u}, \mathbf{u}). \tag{2.2}$$

We see that the poits in $\mathcal{H}^+_{\frac{1}{2}\mathbf{u}}(\mathbf{0}, L)$ are the points that are of closer or equal distance to $\mathbf{0}$ than $\mathbf{u}$.

Proposition 2.1. *The set $\mathcal{H}^+_{\frac{1}{2}\mathbf{u}}(\mathbf{0}, L)$ is a convex set.*

Proposition 2.2. *The intersetion of any number of convex sets is also convex.*

With these preparation we define the Dirichlet-Voronoi region of L around $\mathbf{0}$ as

$$Vor(\mathbf{0}, L) = \bigcap_{\mathbf{u} \in L \setminus \mathbf{0}} \mathcal{H}^+_{\frac{1}{2}\mathbf{u}}(\mathbf{0}, L).$$

This set consists of points that are closer to $\mathbf{0}$ than any other lattice points in L.

Proposition 2.3. *Let L be a lattice in $\mathbb{R}^n$. Then the Dirichlet-Voronoi region of L around $\mathbf{0}$ is convex in $\mathbb{R}^n$.*

3 Basic Theorem

Theorem 3.1. *Let L be a lattice in $\mathbb{R}^n$. Let $Vor(\mathbf{0}, L)$ be the Dirichlet-Voronoi region of L arround $\mathbf{0}$. The quadratic function*

$$f(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n) = \mathbf{x}_1^2 + \mathbf{x}_2^2 + \cdots + \mathbf{x}_n^2$$

that is defined on $Vor(\mathbf{0}, L)$ attain its maximal value at some verteces of $Vor(\mathbf{0}, L)$. We call such verteces deep holes of L.

The problem says that we want to find maximal value of the quadratic function f under linear constraints (2). This is a special case of the quadratic programming problems. The square root of the maximal value in Theorem 3.1 is the covering radius of L, and we denote it by $\rho(L)$.

4 Two Major Trends of problems

We viewed some of the basic references ([7],[43],[46],[48]). The present speaker does not have a chance to read [20]. We may note that there are two major trends in studying the covering radius problems in the class of positive definite lattices.

4.1 First trend

In the dimensions where the reduction theory is well studied the Dirichlet-Voronoi region for a given reduced basis of a lattice L is determined.
In [27] Lagrange determined the conditions of reducedness for the binary positive definite quadratic forms. In [15] Dirichlet determined the conditions of reducedness for the ternary positive definite quadratic forms. After ternary case Minkowski [33] gave a sketch of the reducedness conditions for n-ary forms ($2 \leq n \leq 5$) and in [34] Minkowski gave a sketch of the reducedness conditions for senary forms. In these two articles he did not give full details of the sketch. van der Waerden [52] made explicit the reducedness condions for quaternary quadratic forms. Ryskov [44] worked out the case $n = 5$. Tammela [49] worked out the case $n = 6$, and [50] worked out the case $n = 7$.
A natural step to obtain the Dirichlet-Voronoi region associated with a given lattice L is to start from the reduced basis of L and to attain the Dirichlet-Voronoi region by an appropriate process.
Since a Dirichlet-Voronoi region is a convex polyhedron, a combinatorial type of a Dirichlet-Voronoi region is a set of data consisting of the vertices, the edges, the two-dimensional faces,....

A Table of the combinatorial classification of the Dirichlet-Voronoi region.

n	*number of types*	*contributer*
2	2	
3	5	[16], [9]
4	52	[12], [13], [48], [10]
5	?	?

For a specified n to find the best possible lattice in $\mathbb{R}^n$.
To estimate the efficiency of the lattice covering the notion of the thickness $\theta(L)$ is known.

$$\Theta(L) = \frac{Vol_n(S_{\rho(L)})}{Vol(\mathcal{FP})}.$$

For a fixed n the lattice with smaller $\Theta(L)$ is a better lattice covering.

Remark 1. *If L_2 is similar to L_1 with the similitude t. Then we see that $Vol_n(S_{\rho(L_2)}) = t^n Vol_n(S_{\rho(L_1)})$ and $Vol(\mathcal{FP}(\mathcal{L}\in)) = t^n Vol(\mathcal{FP}(\mathcal{L}_\infty))$ holds. Consequently we have $\Theta(L_1) = \Theta(L_2)$.*

A Table of the best known lattice covering.

n	Θ	*lattice*	*source*
2	1.2092	*hexagonal lattice*	[22]
3	1.4635	$A_3^{\#}$	[4], [1], [18]
4	1.7655	$A_4^{\#}$	[14]
5	2.1243	$A_5^{\#}$	[45]
$n \geq 6$	*unknown*		

4.2 Second Trend

When $n \geq 8$ the reduction theory is not well developed explicitly.
A principal strategy to treat the problem is that (i) to determine the exact shape of of the Dirichlet-Voronoi region of the lattice L, and (ii) to determine the covering radius of L. For specified classes of lattices L the covering radius of $\rho(L)$ and its thickness $\Theta(L)$ are known. The irreducible root lattices and their duals

4.2.1 root lattices and their duals

$A_n(n \geq 1), D_n(n \geq 4), E_6, E_7, E_8$. First appearance of these lattices is described in [24, 25, 26] in the form of the quadtratic forms. The lattice version of the root lattices may be due to v.d. Waerden [51] or Witt [59, 60].

4.2.2 extremal lattices

It is known that in dimensions 8,16,24,32,40,48,56,64,72,80, there exists at least one even unimodular extremal lattice.

4.2.3 uniform lattices

A uniform lattice is a lattice which has a basis consisting of minimal vectors.
A root lattice is a uniform lattice. An even unimodular extremal lattice of dimension 8, (resp.16,24, 32, 48, 72) is uniform.
In [54] Venkov has proved that any even unimodular 32-dimensional extremal lattices is generated by the minimal vectors (norm 4).
In [39] the present speaker has showed that any even unimodular 48-dimensional extremal lattice is generated by the minimal vectors of norm 6.

Remark 2. *The uniformity of the Leech lattice is easily read from the binary code construction of the Leech lattice.*
Although the uniformity of a lattice is known it is not easy to give an explicit minimal norm vector basis. Our present method needs to know the explicit basis of a lattice.

In [23] Kominers has showed that any even unimodular 72-dimensional extremal lattice is generated by the minimal vectors of norm 8.

Remark 3. *At the time of the appearance of [23] the existence of even unimodular 72-dimensional extremal lattice is not known. Three years after this work Nebe [36] showed such a lattice. [23] is a kind of speculation.*

5 Examples

Lemma 5.1. *Let L be an integral lattice in $\mathbb{R}^n$. Suppose that $\mathbf{u}_1 \in \Lambda_{m_1}$ and $\mathbf{u}_2 \in \Lambda_{m_2}$ satisfy $(\mathbf{u}_1, \mathbf{u}_2) = 0$. Then it holds that*

$$\mathcal{H}^+_{\frac{1}{2}\mathbf{u}_1}(\mathbf{0}, L) \cap \mathcal{H}^+_{\frac{1}{2}\mathbf{u}_2}(\mathbf{0}, L) \subset \mathcal{H}^+_{\frac{1}{2}(\mathbf{u}_1+\mathbf{u}_2)}(\mathbf{0}, L).$$

Proof. We put $\mathbf{v} = \mathbf{u}_1 + \mathbf{u}_2$). The defining inequality for $\mathcal{H}^+_{\frac{1}{2}\mathbf{v}}(\mathbf{0}, L)$ is

$$(\mathbf{x}, \mathbf{v}) \leq 1/2(\mathbf{v}, \mathbf{v}).$$

We observe that

$$\begin{aligned} (\mathbf{x}, \mathbf{v}) &= (\mathbf{x}, \mathbf{u}_1) + (\mathbf{x}, \mathbf{u}_2) \\ &\leq \frac{1}{2}(\mathbf{v}, \mathbf{v}) \\ &= \frac{1}{2}(\mathbf{u}_1, \mathbf{u}_1) + \frac{1}{2}(\mathbf{u}_2, \mathbf{u}_2). \end{aligned}$$

If $\mathbf{x} \in \mathcal{H}^+_{\frac{1}{2}\mathbf{u}_1}(\mathbf{0}, L)$ and $\mathbf{x} \in \mathcal{H}^+_{\frac{1}{2}\mathbf{u}_2}(\mathbf{0}, L)$. Then $\mathbf{x} \in \mathcal{H}^+_{\frac{1}{2}(\mathbf{u}_1+\mathbf{u}_2)}(\mathbf{0}, L)$. This is what we should show. □

5.1 D_4 case

Let $D_4 = [\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_3 - \mathbf{e}_4, \mathbf{e}_3 + \mathbf{e}_4]_Z$, where

$$\mathbf{e}_1 = (1, 0, 0, 0), \mathbf{e}_2 = (0, 1, 0, 0), \mathbf{e}_3 = (0, 0, 1, 0), \mathbf{e}_4 = (0, 0, 0, 1).$$

A fundamental parallelepiped $FP^{++++}(D_4)$ is defined by

$$\begin{aligned} &FP^{++++}(D_4) = \\ &\quad \{(x_1, x_2, x_3, x_4) | (x_1, x_2, x_3, x_4) = a_1(\mathbf{e}_1 - \mathbf{e}_2) + a_2(\mathbf{e}_2 - \mathbf{e}_3) + a_3(\mathbf{e}_3 - \mathbf{e}_4) + a_4(\mathbf{e}_3 + \mathbf{e}_4), \\ &\quad 0 \leq a_i \leq 1, a_i \in \mathbb{R}, i = 1, 2, 3, 4\}. \end{aligned}$$

Let

$$\mathcal{D} = \{\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | (\mathbf{x}, \mathbf{u}) \leq \frac{1}{2}(\mathbf{u}, \mathbf{u}) = 1, \mathbf{u} \in \Lambda_2(D_4)\}.$$

The defining inequalities for $\mathcal{D}$ are

$$-1 \leq x_i - x_j \leq 1, -1 \leq x_i + x_j \leq 1, 1 \leq i < j \leq 4.$$

By elementay considerations we find that the vertices of $\mathcal{D}$ are

$$x_1 = \pm\frac{1}{2}, x_2 = \pm\frac{1}{2}, x_3 = \pm\frac{1}{2}, x_4 = \pm\frac{1}{2}, \text{or } x_i = \pm 1, x_j = 0 (j \neq i).$$

Since we have $\frac{1}{2}\sqrt{(\mathbf{v}, \mathbf{v})} \geq 1$ for $\mathbf{v} \in \Lambda_{2m}, m \geq 2$, we conclude that

$$Vor(\mathbf{0}, D_4) = \mathcal{D}.$$

The covering radius of D_4 is 1.

5.2 Leech lattice

$$\begin{aligned} \mid \Lambda_2 \mid &= 0, \\ \mid \Lambda_4 \mid &= 196560, \\ \mid \Lambda_6 \mid &= 16773120, \\ \mid \Lambda_8 \mid &= 398034000. \end{aligned}$$

Proposition 5.2. *Let $\mathcal{L}$ be the Leech lattice and $\Lambda_4 = \Lambda_4(\mathcal{L})$, then we have*

$$\sum_{\mathbf{x} \in \Lambda_4} (\mathbf{x}, \alpha)^2 = 32760(\alpha, \alpha) \tag{5.1}$$

$$\sum_{\mathbf{x} \in \Lambda_4} (\mathbf{x}, \alpha)^4 = 15120(\alpha, \alpha)^2 \tag{5.2}$$

$$\sum_{\mathbf{x} \in \Lambda_4} (\mathbf{x}, \alpha)^6 = 10800(\alpha, \alpha)^3 \tag{5.3}$$

$$\sum_{\mathbf{x} \in \Lambda_4} (\mathbf{x}, \alpha)^8 = 10080(\alpha, \alpha)^4 \tag{5.4}$$

$$\sum_{\mathbf{x} \in \Lambda_4} (\mathbf{x}, \alpha)^{10} = 11340(\alpha, \alpha)^5 \tag{5.5}$$

$$\sum_{\mathbf{x} \in \Lambda_4} (\mathbf{x}, \alpha)^{14} - \frac{91 \cdot (\alpha, \alpha)}{12} \sum_{\mathbf{x} \in \Lambda_4} (\mathbf{x}, \alpha)^{12} = -90090 \cdot (\alpha, \alpha)^7 \tag{5.6}$$

Proposition 5.3. *Let $\mathcal{L}$ be the Leech lattice and $\Lambda_6 = \Lambda_6(\mathcal{L})$, then we have*

$$\sum_{\mathbf{x}\in\Lambda_6} (\mathbf{x}, \alpha)^2 = 4193280(\alpha, \alpha) \tag{5.7}$$

$$\sum_{\mathbf{x}\in\Lambda_6} (\mathbf{x}, \alpha)^4 = 2903040(\alpha, \alpha)^2 \tag{5.8}$$

$$\sum_{\mathbf{x}\in\Lambda_6} (\mathbf{x}, \alpha)^6 = 3110400(\alpha, \alpha)^3 \tag{5.9}$$

$$\sum_{\mathbf{x}\in\Lambda_6} (\mathbf{x}, \alpha)^8 = 4354560(\alpha, \alpha)^4 \tag{5.10}$$

$$\sum_{\mathbf{x}\in\Lambda_6} (\mathbf{x}, \alpha)^{10} = 7348320(\alpha, \alpha)^5 \tag{5.11}$$

5.3 Dirichlet-Voronoi region of the Leech lattice

Theorem 5.4. *Let $\mathcal{L}_{24}$ be the Leech lattice. Then Dirichlet-Voronoi region $Vor(\mathbf{0}, \mathcal{L}_{24})$ of $\mathcal{L}_{24}$ arround $\mathbf{0}$ is determined by*

$$Vor(\mathbf{0}, \mathcal{L}_{24}) = \bigcap_{\mathbf{u}\in\Lambda_4\cup\Lambda_6} \mathcal{H}^+_{\frac{1}{2}\mathbf{u}}(\mathbf{0}, L).$$

Proof. A sketch of the proof.
As the first approximation of Dirichlet-Voronoi region for the Leech lattice we begin with

$$\bigcap_{\mathbf{u}\in\Lambda_4} \mathcal{H}^+_{\frac{1}{2}\mathbf{u}}(\mathbf{0}, L).$$

Take any $\mathbf{v} \in \Lambda_6$. Then we put

$$\alpha = \mathbf{v}, \lambda_k = \#\{\mathbf{u} \in \Lambda_4 | (\mathbf{u}, \mathbf{v}) = k\}.$$

By a simple argument we can show that $-3 \leq k \leq 3$ and by Proposition 5.2 we have the relations

$$\begin{aligned}
2 \cdot 3^2\lambda_3 + 2 \cdot 2^2\lambda_2 + 2 \cdot \lambda_1 &= 32760 \cdot 6 \\
2 \cdot 3^4\lambda_3 + 2 \cdot 2^4\lambda_2 + 2 \cdot \lambda_1 &= 15120 \cdot 6^2 \\
2 \cdot 3^6\lambda_3 + 2 \cdot 2^6\lambda_2 + 2 \cdot \lambda_1 &= 10800 \cdot 6^3
\end{aligned}$$

By solving these equations we have

$$\lambda_3 = 252, \lambda_2 = 12978, \lambda_1 = 44100.$$

We consider the vectors $\mathbf{u} \in \Lambda_4$ which satisfy $(\mathbf{u}, \mathbf{v}) = 3$. Take such a vector $\mathbf{u}$. The angle of intersection θ between $\mathbf{v}$ and $\mathbf{u}$ satifies

$$\cos\theta = \frac{(\mathbf{u}, \mathbf{v})}{\sqrt{(\mathbf{v}, \mathbf{v})} \cdot \sqrt{(\mathbf{u}, \mathbf{u})}} = \frac{3}{2 \cdot \sqrt{6}}.$$

The hyperplane which is perpendicular to the vector $\mathbf{u}$ and intersects with $\mathbf{u}$ at the point $\frac{1}{2}\mathbf{u}$ should meet with the vector $\mathbf{v}$ at $c\mathbf{v}$. We see a geometric relation:

$$\sqrt{(c\mathbf{v}, c\mathbf{v})} \cos\theta = \frac{1}{2}((\mathbf{u}, \mathbf{u}).$$

Thus we have

$$c = \frac{2}{3}.$$

This shows that the point $\frac{1}{2}\mathbf{v}$ is inside of $\bigcap_{\mathbf{u}\in\Lambda_4} \mathcal{H}^+_{\frac{1}{2}\mathbf{u}}(\mathbf{0}, L)$, and the half-space $\mathcal{H}^+_{\frac{1}{2}\mathbf{u}}(\mathbf{0}, L)$ sharpens $\bigcap_{\mathbf{u}\in\Lambda_4} \mathcal{H}^+_{\frac{1}{2}\mathbf{u}}(\mathbf{0}, L)$. Thus the second approximation of the Dirichlet-Voronoi region for the Leech lattice we obtain

$$\bigcap_{\mathbf{u}\in\Lambda_4\cup\Lambda_6} \mathcal{H}^+_{\frac{1}{2}\mathbf{u}}(\mathbf{0}, L).$$

It remains to show that the half spaces $\bigcap_{\mathbf{u}\in\Lambda_{2m}} \mathcal{H}^+_{\frac{1}{2}\mathbf{u}}(\mathbf{0}, L), m \geq 4$ do not affect to

$$Vor(\mathbf{0}, Leech) = \bigcap_{\mathbf{u}\in Leech\setminus\mathbf{0}} \mathcal{H}^+_{\frac{1}{2}\mathbf{u}}(\mathbf{0}, Leech).$$

$\square$

We quote a result in [7], Chap. 22 and Chap. 23.

Theorem 5.5. *The covering radius of the Leech lattice is* $\sqrt{2}$.

Remark 4. *Let* $G = Aut(\mathcal{L}_{24})$ *be the automorphism group of the Leech lattice. Then any element* $\sigma \in G$ *acts on the Dirichlet-Voronoi region of the Leech lattice.*

$$\mathcal{H}^+_{\frac{1}{2}\mathbf{u}}(\mathbf{0}, L) \to \mathcal{H}^+_{\frac{1}{2}\mathbf{u}^\sigma}(\mathbf{0}, L).$$

Thus G *acts also on the set of the deep holes of the Leech lattice. The Dirichlet-Voronoi region has also another kind of holes (shallow holes).*

5.4 Even Unimodular Extremal 32-dimensional Lattices

When $\mathbf{C}$ is a doubly even self-dual binary $[32, 16, 8]$ code and $L(\mathbf{C}) = \mathcal{N}(\mathbf{C})$ is the even unimodular extremal lattice constructed from $\mathbf{C}$ in the previous section, we put $\Lambda_{2k} = \{\mathbf{x} \in L \mid (\mathbf{x}, \mathbf{x}) = 2k\} \quad (k \geq 0)$. The cardinality of the set Λ_{2k} is denoted by $\mid \Lambda_{2k} \mid$. The following cardinalities are well-known:

$$\begin{aligned} \mid \Lambda_6 \mid &= 64757760, \\ \mid \Lambda_8 \mid &= 4844836800. \end{aligned}$$

We are particularly interested in the set $\Lambda_4(L(\mathbf{C}))$. $\Lambda_4 = \Lambda_4(L(\mathbf{C}))$ is a union of six mutually disjoint subsets:

$$\Lambda_4 = \Lambda_{4,1} \cup \Lambda_{4,2} \cup \Lambda_{4,3} \cup \Lambda_{4,4} \cup \Lambda_{4,5} \cup \Lambda_{4,6}, \tag{8.0}$$

defined by

Proposition 5.6. *Let $\mathcal{L}_{32}$ be an even unimodular extremal 32-dimensional lattice and $\Lambda_4 = \Lambda_4(\mathcal{L}_{32})$, then we have*

$$\sum_{\mathbf{x}\in\Lambda_4} (\mathbf{x}, \alpha)^2 = 18360(\alpha, \alpha), \tag{5.12}$$

$$\sum_{\mathbf{x}\in\Lambda_4} (\mathbf{x}, \alpha)^4 = 6480(\alpha, \alpha)^2, \tag{5.13}$$

$$\sum_{\mathbf{x}\in\Lambda_4} (\mathbf{x}, \alpha)^6 = 3600(\alpha, \alpha)^3, \tag{5.14}$$

$$\sum_{\mathbf{x}\in\Lambda_4} (\mathbf{x}, \alpha)^{10} - \frac{15\cdot(\alpha,\alpha)}{4} \sum_{\mathbf{x}\in\Lambda_4} (\mathbf{x}, \alpha)^8 = -7560\cdot(\alpha, \alpha)^5. \tag{5.15}$$

The following statement may be possible to prove. (We have not completed the proof yet.)

Theorem 5.7. *Let $\mathcal{L}_{32}$ be one of even unimodular extremal lattices. Then Dirichlet-Voronoi region $Vor(\mathbf{0}\mathcal{L}_{32}$ of $\mathcal{L}_{32}$ arround $\mathbf{0}$ is determined by*

$$Vor(\mathbf{0}, \mathcal{L}_{32}) = \bigcap_{\mathbf{x}\in\Lambda_4\cup\Lambda_6} \mathcal{H}^+_{\frac{1}{2}\mathbf{u}}(\mathbf{0}, L).$$

Remark 5. *Even if we could prove the above statement it takes much effort to determine the covering radius of $\mathcal{L}_{32}$. At present we face the complex computational obstacles for finding the vertices of the Dirichlet-Voronoi region of $\mathcal{L}_{32}$.*

5.5 48-dimensional Even Unimodular Extremal Lattices

Proposition 5.8. *Let $\mathcal{L}_{48}$ be an even unimodular 48 dimensional extremal lattice , $\Lambda_6 = \Lambda_6(\mathcal{L}_{48})$ and $\boldsymbol{\alpha} \in \mathcal{L}_{48} \bigotimes \mathbb{R}$, then we have*

$$\sum_{\mathbf{x}\in\Lambda_6} (\mathbf{x}, \boldsymbol{\alpha})^2 = 6552000(\boldsymbol{\alpha}, \boldsymbol{\alpha}) \tag{5.16}$$

$$\sum_{\mathbf{x}\in\Lambda_6} (\mathbf{x}, \boldsymbol{\alpha})^4 = 2358720(\boldsymbol{\alpha}, \boldsymbol{\alpha})^2 \tag{5.17}$$

$$\sum_{\mathbf{x}\in\Lambda_6} (\mathbf{x}, \boldsymbol{\alpha})^6 = 1360800(\boldsymbol{\alpha}, \boldsymbol{\alpha})^3 \tag{5.18}$$

(5.19)
$$\sum_{\mathbf{x}\in\Lambda_6}(\mathbf{x},\boldsymbol{\alpha})^8 = 1058400(\boldsymbol{\alpha},\boldsymbol{\alpha})^4$$

(5.20)
$$\sum_{\mathbf{x}\in\Lambda_6}(\mathbf{x},\boldsymbol{\alpha})^{10} = 1020600(\boldsymbol{\alpha},\boldsymbol{\alpha})^5$$

(5.21)
$$\sum_{\mathbf{x}\in\Lambda_6}(\mathbf{x},\boldsymbol{\alpha})^{14} - \frac{91\cdot(\boldsymbol{\alpha},\boldsymbol{\alpha})}{12}\sum_{\mathbf{x}\in\Lambda_6}(\mathbf{x},\boldsymbol{\alpha})^{12} = -7297290\cdot(\boldsymbol{\alpha},\boldsymbol{\alpha})^7$$

Remark 6. *We could make a statement for $\mathcal{L}_{48}$ similar to Theorem 5.7, but it is not the time to circulate it.*

5.6 72-dimensional Even Unimodular Extremal Lattices

Proposition 5.9. *Let $\mathcal{L}_{72}$ be an even unimodular 72 dimensional extremal lattice , $\Lambda_8 = \Lambda_8(\mathcal{L}_{72})$ and $\boldsymbol{\alpha} \in \mathcal{L}_{72}\bigotimes\mathbb{R}$, then we have*

(5.22)
$$\sum_{\mathbf{x}\in\Lambda_8}(\mathbf{x},\boldsymbol{\alpha})^2 = 690908400(\boldsymbol{\alpha},\boldsymbol{\alpha})$$

(5.23)
$$\sum_{\mathbf{x}\in\Lambda_8}(\mathbf{x},\boldsymbol{\alpha})^4 = 224078400(\boldsymbol{\alpha},\boldsymbol{\alpha})^2$$

(5.24)
$$\sum_{\mathbf{x}\in\Lambda_8}(\mathbf{x},\boldsymbol{\alpha})^6 = 117936000(\boldsymbol{\alpha},\boldsymbol{\alpha})^3$$

(5.25)
$$\sum_{\mathbf{x}\in\Lambda_8}(\mathbf{x},\boldsymbol{\alpha})^8 = 84672000(\boldsymbol{\alpha},\boldsymbol{\alpha})^4$$

(5.26)
$$\sum_{\mathbf{x}\in\Lambda_8}(\mathbf{x},\boldsymbol{\alpha})^{10} = 76204800(\boldsymbol{\alpha},\boldsymbol{\alpha})^5$$

(5.27)
$$\sum_{\mathbf{x}\in\Lambda_8}(\mathbf{x},\boldsymbol{\alpha})^{14} - \frac{91\cdot(\boldsymbol{\alpha},\boldsymbol{\alpha})}{12}\sum_{\mathbf{x}\in\Lambda_6}(\mathbf{x},\boldsymbol{\alpha})^{12} = -518918400\cdot(\boldsymbol{\alpha},\boldsymbol{\alpha})^7$$

6 Problems

- For many of Niemeier lattices the Dirichlet-Voronoi regions, covering radii are not known.
- For $\mathcal{L}_{48}$ and $\mathcal{L}_{72}$ we know that both lattices have minimal basis. But we do not know explicit forms of the basis. For this reason we can not know the precise shape of the Dirichlet-Voronoi regions of these two lattices.
- When the minimal basis has the different norms (even they are reduced). The determination of the covering radius of the lattice would be much hard.
- For the class of odd unimodular lattices not many results are obtained.

7 Appendix

Let S_r be a sphere of radius r in the n-dimensional Euclidean space $\mathbb{R}^n$. Then the volume of S_r is given by

$$Vol_n(S_r) = \frac{\pi^{n/2} r^n}{\Gamma(\frac{n}{2}+1)}.$$

n	$Vol_n(S_r)$
0	1
1	$2r$
2	πr^2
3	$\frac{4\pi}{3} r^3$
4	$\frac{\pi^2}{2} r^4$
5	$\frac{8\pi^2}{15} r^5$
6	$\frac{\pi^3}{6} r^6$
7	$\frac{16\pi^3}{105} r^7$

n	$Vol_n(S_r)$
8	$\frac{\pi^4}{24} r^8$
9	$\frac{32\pi^4}{945} r^9$
10	$\frac{\pi^5}{120} r^{10}$
24	$\frac{\pi^{12}}{12!} r^{24}$
32	$\frac{\pi^{16}}{16!} r^{32}$
48	$\frac{\pi^{24}}{24!} r^{48}$

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